TRINITY COLLEGE

Year 12 Mathematics Specialist 3/4
Test 4 2022

Weighting: 7%

Scientific Calculator Assumed
Integration

STUDENT'S NAME Solutions [PRESSER]

DATE: Monday 25 July **TIME:** 50 minutes **MARKS**: 50

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three **Scientific** calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (6 marks)

Using the given substitutions or otherwise, determine the following integrals

(a) $\frac{1}{2}\int 2(\cos x)\cos(2\sin x)\,dx$ let $y = 2\sin x$ $\Rightarrow dy = 2\cos x\,dx$ [3]

$= \frac{1}{2}\int 2\cancel{\cos x}\cos(2\sin x)\frac{dy}{2\cancel{\cos x}}$

$= \frac{1}{2}\int \cos y\,dy$

$= \frac{1}{2}\sin y + c \quad = \underline{\underline{\frac{1}{2}\sin(2\sin x) + C}}$

✓sub
✓ integration
✓ terms of x

(b) $\int \frac{\sin 2x}{4+3\cos^2 x}\,dx$ let $y = 4 + 3\cos^2 x$ $\Rightarrow dy = -6\cos x\sin x\,dx$ [3]

$= \int \frac{2\sin x\cos x}{y}\cdot\frac{dy}{-6\cos x\sin x}$

$= -\frac{1}{3}\int \frac{1}{y}\,dy$

$= -\frac{1}{3}\ln|y| + c$

$= -\frac{1}{3}\ln|4+3\cos^2 x| + c$

✓ sub
✓ integration
✓ terms of x

2. (11 marks)

(a) Determine $\int \frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}\,dx$ using the integral: $\int \frac{f'(x)}{f(x)}\,dx = \ln |f(x)| + c$ [1]

$$= \ln|3\sin x + 4\cos x| + c$$

✓ ans

(b) For the equation $\sin x = a(3\sin x + 4\cos x) + b(3\cos x - 4\sin x)$.

(i) two simultaneous equations can be formed. Given that one of the equations is : $3a - 4b = 1$. Determine a second equation and solve for a and b. [3]

$$\begin{cases} 4a + 3b = 0 \\ 3a - 4b = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 16a + 12b = 0 \\ 9a - 12b = 3 \end{cases}$$

$$\Rightarrow 25a = 3$$

$$a = 3/25$$

$$4b = 3a - 1$$

$$4b = -\frac{16}{25}$$

$$b = -\frac{4}{25}$$

✓ 2nd eqn

✓ a

✓ b

(ii) hence, using (b)(i), show that: [3]

$$\frac{\sin x}{3\sin x + 4\cos x} = \frac{3}{25}\left(1 - \frac{4}{3}\left(\frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}\right)\right)$$

From b $\sin x = \frac{3}{25}(3\sin x + 4\cos x) - \frac{4}{25}(3\cos x - 4\sin x)$

$$\Rightarrow \frac{\sin x}{3\sin x + 4\cos x} = \frac{3}{25} - \frac{4}{25}\left(\frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}\right)$$

$$= \frac{3}{25}\left[1 - \frac{4}{3}\left(\frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}\right)\right]$$

✓ exp (using b)

✓ $\div\ 3\sin x + 4\cos x$

✓ factorise

(c) Hence, show $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{3\sin x + 4\cos x}\,dx = \frac{1}{50}\left(3\pi + 8\ln\left(\frac{4}{3}\right)\right)$ [4]

$$= \frac{3}{25}\int_0^{\pi/2} 1 - \frac{4}{3}\left(\frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}\right) dx$$ from 6 ii

$$= \frac{3}{25}\left[x - \frac{4}{3}\ln\left|3\sin x + 4\cos x\right|\right]_0^{\pi/2}$$

$$= \frac{3}{25}\left[\left(\frac{\pi}{2} - \frac{4}{3}\ln 3\right) - \left(0 - \frac{4}{3}\ln 4\right)\right]$$

$$= \frac{3}{25}\left[\frac{\pi}{2} + \frac{4}{3}\ln\left(\frac{4}{3}\right)\right]$$

$$= \frac{1}{50}\left[3\pi + 8\ln\frac{4}{3}\right]$$

✓ sub

✓ integration

✓ sub in boundaries

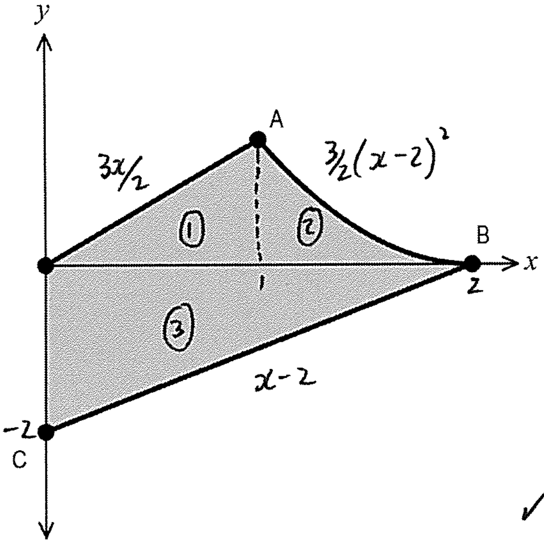
✓ factorise

3. (8 marks)

Consider the following graphs of the functions $y = \frac{3x}{2}$, $y = \frac{3}{2}(x-2)^2$ and $y = x - 2$.

Determine

(a) The coordinates of the points A, B and C. [3]



$$\frac{\cancel{3}x}{\cancel{2}} = \frac{\cancel{3}(x-2)^2}{\cancel{2}} \Rightarrow x = 1, 4$$

$$0 = x^2 - 3x + 4$$

So $A = (1, 3/2)$

$B = (2, 0)$ x-int tp

$C = (0, -2)$ y-int

✓ A
✓ B
✓ C

(b) An integral, or a combination of integrals that will give the area of the shaded region.
(Do not evaluate) [2]

$$\text{Area} = \int_0^1 \frac{3x}{2}\,dx + \int_1^2 \frac{3}{2}(x-2)^2\,dx - \int_0^2 x-2\,dx$$

OR $$\text{Area} = \frac{3}{4} + \int_1^2 \frac{3}{2}(x-2)^2\,dx + 2$$

✓ +ve Area
✓ −ve Area

(c) The exact area of the shaded region. [3]

$$\text{Area} = \frac{3}{4} + 2 + \frac{3}{2}\int_1^2 (x-2)^2\,dx$$

$$= \frac{11}{4} + \frac{3}{2}\cdot\frac{1}{3}\Big[(x-2)^3\Big]_1^2$$

$$= \frac{11}{4} + \frac{1}{2}\Big[(0) - (-1)\Big]$$

$$= \frac{13}{4} \text{ units}^2$$

✓ int
✓ sub boundaries
✓ ans

4. (6 marks)

Using the substitution $x = \sin^2\theta$, show that $\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\, dx = \frac{\pi}{4} - \frac{1}{2}$

$$= \int_0^{\pi/4} \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\; 2\sin\theta\cos\theta \, d\theta$$

$$x = \sin^2\theta$$

$$dx = 2\sin\theta\cos\theta \, d\theta$$

$$= \int_0^{\pi/4} \frac{\sin\theta}{\cancel{\cos\theta}}\; 2\sin\theta\cancel{\cos\theta}\, d\theta$$

$$x = \tfrac{1}{2} \qquad \pm\tfrac{1}{\sqrt{2}} = \sin\theta$$

$$\Rightarrow \quad \theta = \pi/4$$

$$= \int_0^{\pi/4} 1 - \cos 2\theta \; d\theta$$

$$x = 0 \qquad 0 = \sin\theta$$

$$\Rightarrow \quad \theta = 0$$

$$= \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= \left(\pi/4 - \tfrac{1}{2}\right) - (0 - 0)$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

✓ boundaries for θ

✓ sub

✓ use $2\sin x\cos x$

✓ simplify trig using $2\sin^2\theta = 1 - \cos 2\theta$

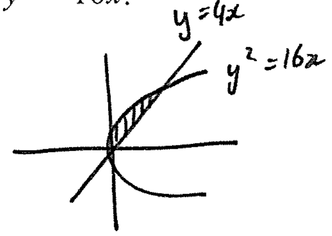
✓ Integrate

✓ sub boundaries

5. (12 marks)

(a) Determine the exact area of the finite region enclosed by the line $y = 4x$ and the curve $y^2 = 16x$. [4]

✓ intersection
✓ expression
✓ integration
✓ sub & ans



$$16x = 16x^2$$
$$\Rightarrow x(x-1) = 0$$
$$\Rightarrow x = 0 \text{ or } 1$$

$$\text{Area} = \int_0^1 \sqrt{16x} - 4x \, dx$$
$$= \int_0^1 4x^{1/2} - 4x \, dx$$
$$= 4\left[\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_0^1$$
$$= \frac{2}{3} \text{ units}^2$$

(b) Determine the exact volume generated when this region is rotated completely about

(i) the x-axis [4]

$$V_x = \pi \int_0^1 16x - 16x^2 \, dx$$
$$= 16\pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$
$$= 16\pi\left[\left(\frac{1}{2} - \frac{1}{3}\right) - (0 - 0)\right]$$
$$= \frac{8\pi}{3} \text{ units}^3$$

✓ expression
✓ integral
✓ sub
✓ ans

(ii) the y-axis [4]

$$V_y = \pi \int_0^4 \frac{y^2}{16} - \frac{y^4}{256} \, dy$$
$$= \frac{\pi}{16}\left[\frac{y^3}{3} - \frac{y^5}{80}\right]_0^4$$
$$= \frac{\pi}{6}\left[\left(\frac{64}{3} - \frac{64}{5}\right) - (0 - 0)\right]$$
$$= \frac{8\pi}{15} \text{ units}^3$$

OR

$$V_{shell} = 2\pi \int_0^1 x(4x^{1/2} - 4x) \, dx$$
$$\Rightarrow V = 2\pi \int_0^1 4x^{3/2} - 4x^2 \, dx$$
$$= 8\pi\left[\frac{2}{5}x^{5/2} - \frac{x^3}{3}\right]_0^1$$
$$= 8\pi\left[\left(\frac{2}{5} - \frac{1}{3}\right) - (0 - 0)\right]$$
$$= \frac{8\pi}{15} \text{ units}^3$$

6. (7 marks)

(a) Using the substitution $u = \sin x$, show that $\int \frac{\cos x}{3 + \cos^2 x}\, dx = \int \frac{du}{4 - u^2}$ [3]

$$\Rightarrow I = \int \frac{\cos x}{3+(1-\sin^2 x)}\, dx$$

$u = \sin x$

$du = \cos x\, dx$

$$= \int \frac{\cancel{\cos x}}{4-u^2} \cdot \frac{du}{\cancel{\cos x}}$$

$$= \int \frac{du}{4-u^2}$$

✓ diff

✓ sub

✓ factorise

(b) Hence, using partial fractions, show that $\int_0^{\frac{\pi}{2}} \frac{\cos x}{3 + \cos^2 x}\, dx = \frac{1}{4} \ln 3$ [4]

From a

$x = \pi/2$, $u = 1$

$x = 0$, $u = 0$

$$I = \int_0^1 \frac{du}{4-u^2}$$

$$= \int_0^1 \frac{1}{(2+u)(2-u)}\, du$$

let

$$\frac{1}{(2+u)(2-u)} = \frac{A}{2+u} + \frac{B}{2-u}$$

$\Rightarrow 1 = A(2-u) + B(2+u)$

$u = 2 \Rightarrow 1 = 4B$

$u = -2 \Rightarrow 1 = 4A$

$$= \frac{1}{4} \int_0^1 \frac{1}{2+u} + \frac{1}{2-u}\, du$$

$$= \frac{1}{4} \Big[\ln|2+u| - \ln|2-u| \Big]_0^1$$

$$= \frac{1}{4} \Big[(\ln 3 - \ln 1) - (\ln 2 - \ln 2) \Big]$$

$$= \frac{1}{4} \ln 3$$

✓ boundaries

✓ partial fractions

✓ integration

✓ substitution